

Nonlinear dynamics of relativistic strings moving in Minkowski space-time

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Aims

- ♣ **Solution formula for**
 - Time-like extremal surfaces with singular points
 - The motion of relativistic strings
- ♣ **Time periodicity of the motion of closed strings**
- ♣ **Characteristic-quadrilateral identity**

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Outline

- ◇ General framework
- ◇ The equations for the motion of relativistic strings
- ◇ Time-like extremal surfaces with singular points
- ◇ Solution formula of the motion
- ◇ Characteristic-quadrilateral identity
- ◇ Numerical analysis and topological singularities
- ◇ Time periodicity of the motion of closed strings
- ◇ Initial-boundary value problem for finite strings
- ◇ Open problems



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1. General framework

♠ \mathcal{M} : m -D manifold with coordinates $(\varphi^1, \dots, \varphi^m)$

♠ (\mathcal{N}, g) : d -D (curved) Lorentz manifold with $(x^0, x^1, \dots, x^{d-1})$

♠ The action for \mathcal{M} moving in \mathcal{N} :

$$\mathcal{A}[x] = \int \sqrt{G} d\varphi^0 d\varphi^1 \dots d\varphi^m,$$

where G and the induced world-volume metric $G_{\alpha\beta}$:

$$G = |\det(G_{\alpha\beta})|,$$

$$G_{\alpha\beta} = x^\mu_{,\alpha} x^\nu_{,\beta} g_{\mu\nu}(x).$$

◇ Einstein summation convention

◇ $\alpha, \beta, \dots = 0, 1, \dots, m; \mu, \nu, \dots = 0, 1, \dots, d-1$

◇ $x^\mu_{,\alpha}$ denotes $\partial x^\mu / \partial \varphi^\alpha$

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★ Time-like assumption: $\det(G_{\alpha\beta}) < 0$

- **Time-like** in mathematics \iff **Causality** in physics

★ The resulting field equations

$$\frac{1}{\sqrt{G}} \left(\sqrt{G} G^{\alpha\beta} x^\mu_{,\alpha} \right)_{,\beta} + G^{\alpha\beta} x^\nu_{,\alpha} x^\rho_{,\beta} \Gamma^\mu_{\nu\rho}(x) = 0$$

where

$(G^{\alpha\beta})$: the inverse matrix of $(G_{\alpha\beta})$

$\Gamma^\mu_{\nu\rho}$: the Christoffel symbols of the metric g
(which vanish if \mathcal{N} is the flat Minkowski space).



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♣ Rich geometric properties: e.g., they are invariant under both arbitrary reparametrizations $\mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R} \times \mathcal{M}$ of the world-volume and isometries $\mathcal{N} \rightarrow \mathcal{N}$.

♣ Classical string theory, the theory of membranes

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2. The equations for relativistic strings

Let (t, x_1, \dots, x_n) be points in \mathbb{R}^{1+n} :

$$ds^2 = \sum_{i=1}^n dx_i^2 - dt^2.$$

A surface \mathcal{S} takes the form

$$x_i = x_i(t, \theta) \quad (i = 1, \dots, n).$$

The Lorentz metric

$$ds^2 = (dt, d\theta) M (dt, d\theta)^T,$$

where

$$M = \begin{pmatrix} |x_t|^2 - 1 & \langle x_t, x_\theta \rangle \\ \langle x_t, x_\theta \rangle & |x_\theta|^2 \end{pmatrix},$$

in which $x = (x_1, \dots, x_n)$ and

$$\langle x_t, x_\theta \rangle = \sum_{i=1}^n x_{i,t} x_{i,\theta}, \quad |x_t|^2 = \langle x_t, x_t \rangle, \quad |x_\theta|^2 = \langle x_\theta, x_\theta \rangle.$$

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Assumption

The surface \mathcal{S} is time-like:

$$\det M < 0,$$

namely,

$$\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2 > 0$$

♣ Time-like in mathematics \Longleftrightarrow Causality in physics



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Time-like, Space-like, Light-like

(1) Vector $\vec{V} \in \mathbb{R}^{1+n}$

- Vanishing vector
- Non-vanishing vector \vec{V}

Time-like: $\alpha \in [0, \pi/4)$

Space-like: $\alpha \in (\pi/4, \pi/2]$

Light-like: $\alpha = \pi/4$

$$\alpha = \angle(\vec{V}, \vec{t})$$



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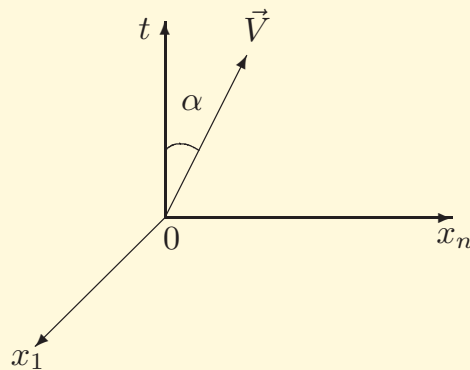


Figure 1: Time-like, Space-like, Light-like vectors



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(2) \mathcal{S} : a surface in \mathbb{R}^{1+n} ; $P \in \mathcal{S}$; normal vector \vec{n}_p

- **\mathcal{S} is time-like (resp. space-like, light-like) at P :**
if \vec{n}_p is space-like (resp. time-like, light-like).
- **\mathcal{S} is time-like (resp. space-like, light-like):**
if it is time-like (resp. space-like, light-like) at every
point $P \in \mathcal{S}$.
- **$P \in \mathcal{S}$ is a singular point: $\vec{n}_p = 0$**

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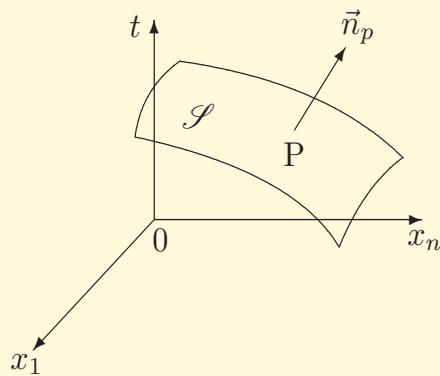


Figure 2: Time-like, Space-like, Light-like surfaces



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The area element of \mathcal{S} is

$$d\mathcal{A} = \sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2} dt d\theta$$

– Nambu-Goto Action in classical string theory.

Definition Surface \mathcal{S} is called to be extremal, if

$$x = x(t, \theta)$$

is a critical point of the area functional

$$\mathcal{A} = \iint \sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2} dt d\theta$$

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The Euler-Lagrange equation

$$\left(\frac{|x_\theta|^2 x_t - \langle x_t, x_\theta \rangle x_\theta}{\sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}} \right)_t - \left(\frac{\langle x_t, x_\theta \rangle x_t - (|x_t|^2 - 1)x_\theta}{\sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}} \right)_\theta = 0 \quad (1)$$

Proposition The equation (1) is equivalent to

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta \rangle x_{t\theta} + (|x_t|^2 - 1)x_{\theta\theta} = 0 \quad (2)$$

for smooth solutions.

♣ In particular, taking $\theta = x_1$ and $n = 2$, (2) is nothing but the classical Born-Infeld equation.


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♣ (2) contains n nonlinear PDEs of second order. These equations show that the surface is extremal if and only if its **mean curvature vector vanishes**.

♣ Although in the process of deriving (2) we assume that the surface is time-like, these equations themselves do not need this assumption.

♣ The metric is Lorentzian (resp. Riemannian), if the surface is time-like (resp. space-like). The corresponding equations are hyperbolic (resp. elliptic) if the surface is time-like (resp. space-like). A connected surface is of mixed type if it contains both a time-like part and a space-like part simultaneously. In this case, the equations (2) are also of mixed type.



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♣ Known results

- Calabi (1970); Cheng and Yau (1976): **Space-like**
- Barbashov, Nesterenko and Chervyakov (CMP, 1982);
- Milnor (1990, Michigan Math. J.);
- Kong (2004, Europhys Lett.);
- Kong, Sun and Yi Zhou (2005, J. Math. Phys.);
- Liu and Yi Zhou (2006):
Time-like surfaces
- Kong, Zhang and Zhou (2006, Comm. Math. Phys.):
Time-like surfaces with singular points
- Gu (1985, 1990, 1994): **Extremal surfaces of mixed type**



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♣ Motivation

Green, Schwarz, Witten (Superstring theory): There is a conformal mapping such that the equations become linear wave equations

Conformal mapping?

♣ Aim

A complete analysis on nonlinear dynamics of relativistic strings moving in \mathbb{R}^{1+n}

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3. Time-like extremal surface with singular points

Let

$$u = x_t, \quad v = x_\theta.$$

Then

$$\begin{cases} u_t - \frac{2\langle u, v \rangle}{|v|^2} u_\theta + \frac{|u|^2 - 1}{|v|^2} v_\theta = 0, \\ v_t - u_\theta = 0. \end{cases}$$

Setting

$$U = (u, v)^T,$$

we have

$$U_t + A(U)U_\theta = 0, \tag{3}$$

where

$$A(U) = \begin{bmatrix} -\frac{2\langle u, v \rangle}{|v|^2} I_{n \times n} & \frac{|u|^2 - 1}{|v|^2} I_{n \times n} \\ -I_{n \times n} & 0 \end{bmatrix}.$$



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Property 1 Under the time-like assumption, (3) is a non-strictly hyperbolic system with two n -constant multiple eigenvalues:

$$\lambda_1 \equiv \cdots \equiv \lambda_n = \lambda_-, \quad \lambda_{n+1} \equiv \cdots \equiv \lambda_{2n} = \lambda_+,$$

where

$$\lambda_{\pm} = \frac{1}{|v|^2} \left(-\langle u, v \rangle \pm \sqrt{\langle u, v \rangle^2 - (|u|^2 - 1)|v|^2} \right).$$

Property 2 λ_{\pm} are linearly degenerate.

Property 3 System (3) is rich in the sense of Serre.

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♣ Riemann invariants

$$R_i = u_i + \lambda_- v_i, \quad R_{i+n} = u_i + \lambda_+ v_i \quad (i = 1, \dots, n).$$

They satisfy

$$\frac{\partial R_i}{\partial t} + \lambda_+ \frac{\partial R_i}{\partial x} = 0, \quad \frac{\partial R_{i+n}}{\partial t} + \lambda_- \frac{\partial R_{i+n}}{\partial x} = 0.$$

♣ Riemann invariants λ_{\pm} :

$$\frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial x} = 0, \quad \frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial x} = 0.$$

- Good gauge: linearizes the equations (1) or (2)

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Consider the Cauchy problem for (1) with

$$t = 0 : \quad x = p(\theta) \in C^2, \quad x_t = q(\theta) \in C^1.$$

Introduce the initial characteristic propagation speeds:

$$\Lambda_{\pm}(\theta) = \frac{-\langle q(\theta), p'(\theta) \rangle \pm \sqrt{\langle q(\theta), p'(\theta) \rangle^2 - (|q(\theta)|^2 - 1)|p'(\theta)|^2}}{|p'(\theta)|^2}.$$

Theorem (Global existence) Suppose that there exist constants Λ_* and Λ^* such that

$$\Lambda_* \leq \Lambda_{\pm}(\theta) \leq \Lambda^* \quad \text{and} \quad \Lambda_+(\theta) > \Lambda_-(\theta), \quad \forall \theta \in \mathbb{R}. \quad (4a)$$

Then the Cauchy problem admits a unique global C^2 solution $x = x(t, \theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, if and only if, for every fixed $\theta_2 \in \mathbb{R}$

$$\Lambda_-(\theta_1) < \Lambda_+(\theta_2), \quad \forall \theta_1 \in (-\infty, \theta_2). \quad (4b)$$

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Moreover, under the assumptions (4)-(5), the global C^2 solution $x = x(t, \theta)$ satisfies that, for arbitrary fixed $(t, \theta) \in \mathbb{R}^+ \times \mathbb{R}$, either (t, θ) is a singular point, i.e.,

$$x_\theta(t, \theta) = 0$$

or, the surface is time-like at (t, θ) , i.e.,

$$\mathcal{L}(t, \theta) > 0, \tag{5}$$

where

$$\mathcal{L}(t, \theta) = \langle x_t(t, \theta), x_\theta(t, \theta) \rangle^2 - (|x_t(t, \theta)|^2 - 1)|x_\theta(t, \theta)|^2.$$

♣ See Kong, Zhang and Zhou (CMP, 2006)

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♣ **Remarks:**

- $(4a)_1$: **Limited initial characteristic propagation speeds**
- $(4a)_2$: **Time-like initial velocity**
- $(4b)$: **Causality law holds in the motion process**
- (5) : **Time-like surfaces with singular points**
- If $(4b)$ is not satisfied, then the surface is no longer time-like and changes its type, for example, from the time-like type to the space-like type.
- **Similar results for the mixed initial-boundary value problems with Dirichlet boundary conditions or Neumann boundary conditions**



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4. Solution formula of the motion

Step 1 Cauchy problem

$$\begin{cases} \frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial \theta} = 0, & \frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial \theta} = 0, \\ t = 0 : \lambda_{\pm} = \Lambda_{\pm}(\theta). \end{cases} \quad (6)$$

- Define

$$\rho(\theta) = \int_0^\theta \frac{2}{\Lambda_+(\xi) - \Lambda_-(\xi)} d\xi,$$
$$\Theta(t, \sigma) = \frac{1}{2} \int_0^{\sigma+t} \Lambda_+(\varrho(\xi)) d\xi - \frac{1}{2} \int_0^{\sigma-t} \Lambda_-(\varrho(\xi)) d\xi.$$

Let $\theta = \varrho(\sigma)$ be the inverse function of $\sigma = \rho(\theta)$ and $\sigma = \Phi(t, \theta)$ be the inverse function of $\theta = \Theta(t, \sigma)$.

- The solution of (6)

$$\lambda_{\pm}(t, \theta) = \Lambda_{\pm}(\varrho(\Phi(t, \theta) \pm t)).$$

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Step 2 Consider the Cauchy problem

$$\begin{cases} \frac{\partial R_i}{\partial t} + \lambda_+(t, \theta) \frac{\partial R_i}{\partial \theta} = 0, \\ \frac{\partial R_{i+n}}{\partial t} + \lambda_-(t, \theta) \frac{\partial R_{i+n}}{\partial \theta} = 0 \end{cases} \quad (i = 1, \dots, n)$$

with the initial data

$$t = 0 : \begin{cases} R_i = q_i(\theta) + \Lambda_-(\theta) p'_i(\theta) \triangleq R_i^0(\theta), \\ R_{i+n} = q_i(\theta) + \Lambda_+(\theta) p'_i(\theta) \triangleq R_{i+n}^0(\theta). \end{cases}$$

The solution of the Cauchy problem

$$\begin{cases} R_i(t, \theta) = R_i^0(\varrho(\Phi(t, \theta) - t)), \\ R_{i+n}(t, \theta) = R_{i+n}^0(\varrho(\Phi(t, \theta) + t)) \end{cases} \quad (i = 1, \dots, n).$$

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Step 3 The solution of (3)

$$\begin{cases} u_i = \frac{\lambda_+(t, \theta) R_i(t, \theta) - \lambda_-(t, \theta) R_{i+n}(t, \theta)}{\lambda_+(t, \theta) - \lambda_-(t, \theta)}, \\ v_i = \frac{R_{i+n}(t, \theta) - R_i(t, \theta)}{\lambda_+(t, \theta) - \lambda_-(t, \theta)}. \end{cases}$$

Step 4 The solution of (2)

$$x(t, \theta) = p(\theta) + \int_0^t u(s, \theta) ds.$$

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Step 5 Introduce characteristic coordinates $\theta_{\pm}(t, \theta)$

$$t = \int_{\theta_-}^{\theta_+} \frac{1}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta,$$
$$\theta = \int_0^{\theta_+} \frac{\Lambda_+(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta - \int_0^{\theta_-} \frac{\Lambda_-(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta.$$

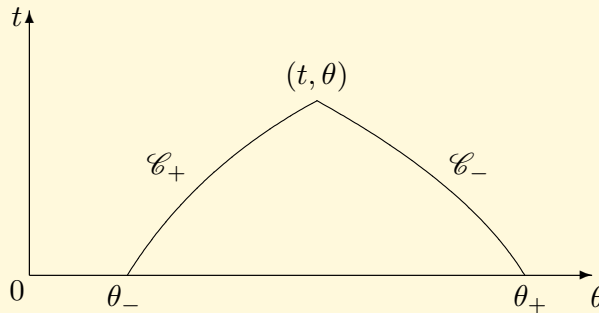


Figure 3: Geometric meaning of (θ_-, θ_+)



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By means of $\theta_{\pm}(t, \theta)$, we have

$$x_i(t, \theta) = \frac{p_i(\theta_+) + p_i(\theta_-)}{2} + \frac{1}{2} \int_{\theta_-}^{\theta_+} \mathcal{Q}_i(\zeta) d\zeta$$

where

$$\mathcal{Q}_i(\zeta) = \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p'_i(\zeta)}{\sqrt{\langle q(\zeta), p'(\zeta) \rangle^2 - (|q(\zeta)|^2 - 1) |p'(\zeta)|^2}}$$

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Remarks

- Solution formula of d'Alembert's type
- Advantages:
 - (a) The trajectory of a point θ in the string: $\theta = \theta(t)$
 - (b) Numerical analysis
 - (c) Time periodicity of the motion of closed strings



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Summary

Original equations



Quasilinear hyperbolic system for $(u, v) = (x_t, x_\theta)$



Properties



Gauge λ_\pm and Riemann invariants R_i ($i = 1, \dots, 2n$)



Solving the gauge λ_\pm



A linear system for R_i ($i = 1, \dots, 2n$)



$(u, v) = (x_t, x_\theta)$ and then $x(t, \theta)$



Introduce characteristic coordinates $\theta_\pm(t, \theta)$



Solution formula of d'Alembert's type



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5. Characteristic-quadrilateral identity

Let Ω be a region in $\mathbb{R}^+ \times \mathbb{R}$, and $x = x(t, \theta)$ be a smooth solution of the equation (1) in this region, which corresponds to a piece of a time-like surface. Let A, B, C and D be points in Ω , and assume that A and B (resp. D and C) are connected by a λ_+ -characteristic and A and D (resp. B and C) are connected by a λ_- -characteristic.

Characteristic-quadrilateral identity

$$x(A) + x(C) = x(B) + x(D)$$

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Idea of proof

Let (t_A, θ_A) be coordinates of the point A , etc.

Special case: $t_B = t_D \triangleq t_0$

Since $x = x(t, \theta)$ is a smooth solution of the equation (1), we have

$$x(t_0, \theta) \triangleq p(\theta), \quad x_t(t_0, \theta) \triangleq q(\theta), \quad \forall \theta \in [\theta_B, \theta_D].$$

Then

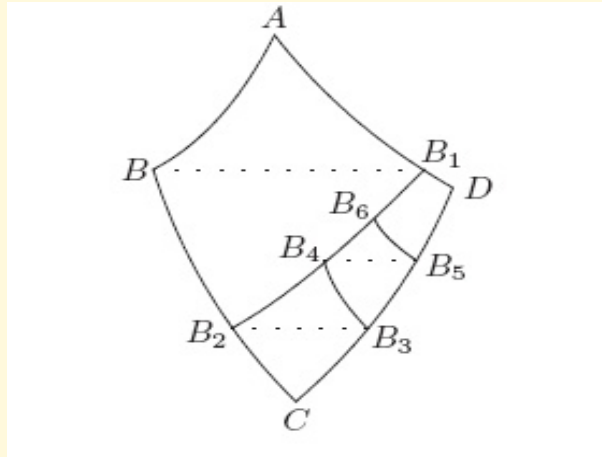
$$\begin{cases} x_i(A) = \frac{p_i(\theta_B) + p_i(\theta_D)}{2} + \frac{1}{2} \int_{\theta_B}^{\theta_D} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p'_i(\zeta)}{\sqrt{\langle q(\zeta), p'(\zeta) \rangle^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta \\ x_i(C) = \frac{p_i(\theta_B) + p_i(\theta_D)}{2} + \frac{1}{2} \int_{\theta_D}^{\theta_B} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p'_i(\zeta)}{\sqrt{\langle q(\zeta), p'(\zeta) \rangle^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta \end{cases}$$

Summing up these equations gives

$$x_i(A) + x_i(C) = p_i(\theta_B) + p_i(\theta_D) = x_i(B) + x_i(D).$$

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General case: $t_B \neq t_D$



Case 1: Finite steps

Similar to the special case



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Case 2: Infinite steps

Continue to divide the resulting characteristic quadrilateral, and denote the resulting characteristic quadrilateral after n step by $\tilde{A}_n \tilde{B}_n \tilde{C}_n D$ (in which \tilde{A}_n is the highest point, and \tilde{C}_n is the lowest one). Noting the boundedness of the characteristics λ_{\pm} ,

$$\tilde{A}_n, \tilde{B}_n, \tilde{C}_n \longrightarrow D \quad \text{as } n \rightarrow \infty.$$

Thus,

$$\left\{ \begin{array}{lcl} x(A) + x(B_2) & = & x(B) + x(B_1), \\ x(B_4) + x(C) & = & x(B_2) + x(B_3), \\ \dots\dots\dots \\ x(\tilde{A}_n) + x(\tilde{C}_n) & = & x(\tilde{B}_n) + x(D) + \\ & & [x(\tilde{A}_n) + x(\tilde{C}_n) - x(\tilde{B}_n) - x(D)]. \end{array} \right.$$

Summing up them leads to

$$\begin{aligned} x(A) + x(C) &= x(B) + x(D) + [x(\tilde{A}_n) + x(\tilde{C}_n) - x(\tilde{B}_n) - x(D)] \\ &\rightarrow x(B) + x(D) \end{aligned}$$

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6. Numerical analysis and topological singularity

Figures 4-6 show the motion of a unit circle with

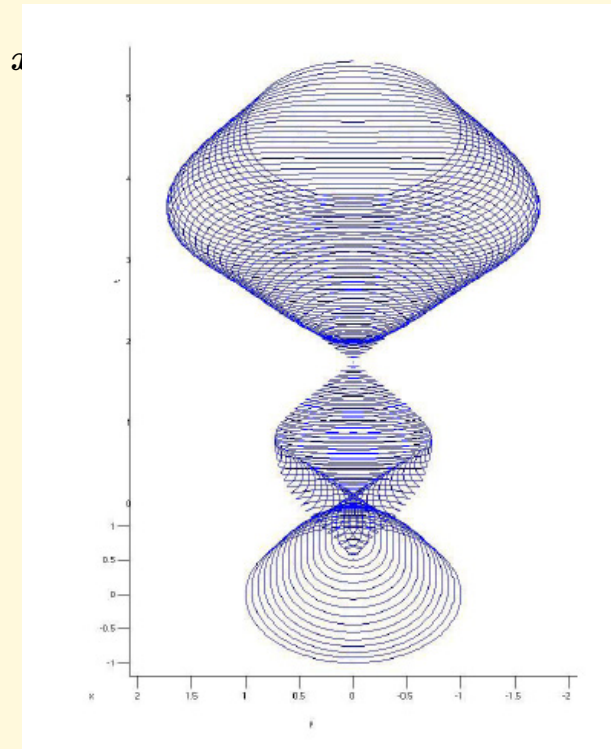


Figure 4: The extremal surface in the Minkowski space \mathbb{R}^{1+2}



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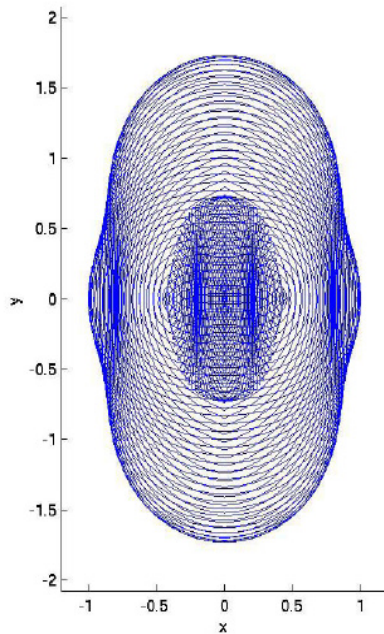


Figure 5: The projection of extremal surface in (x, y) -plane



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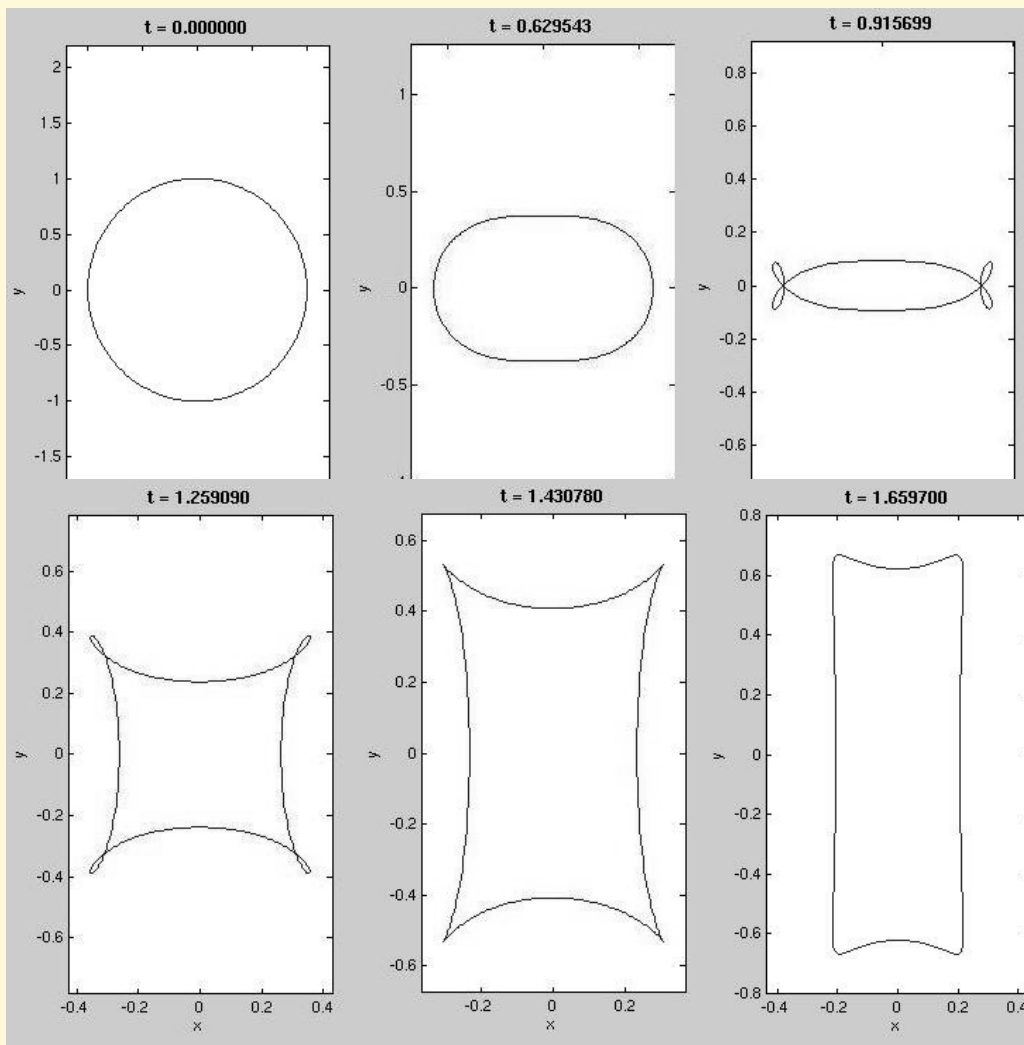
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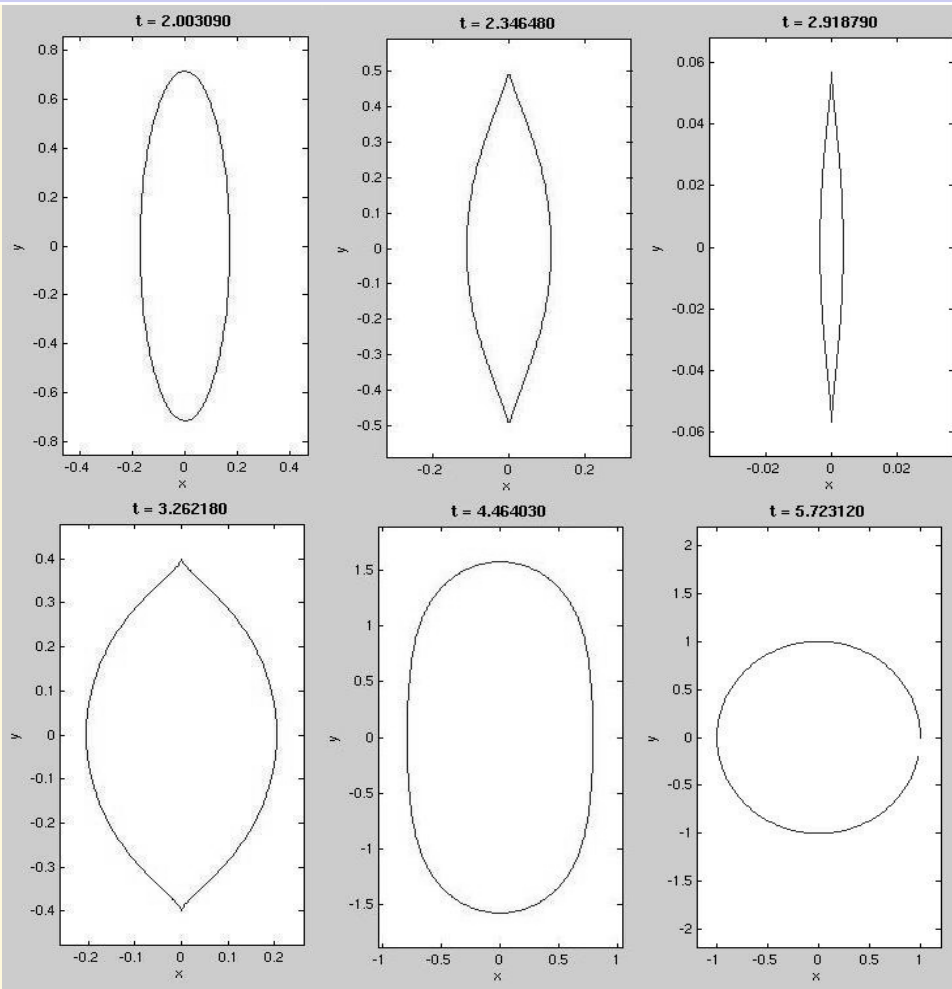


Figure 6: The dynamics of the closed string moving in \mathbb{R}^{1+2}



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Figures 7-9 show the motion of a unit circle with

$$x_t = (0, -0.99 \sin(8\theta)).$$

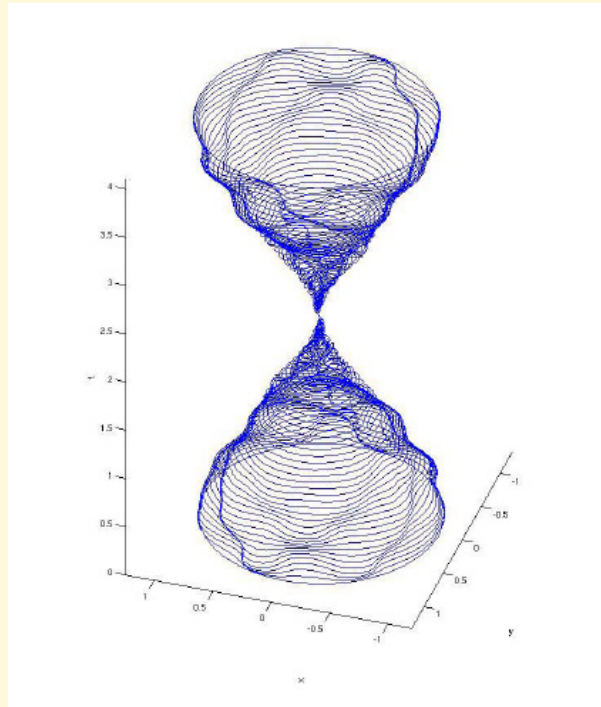


Figure 7: The extremal surface in the Minkowski space \mathbb{R}^{1+2}



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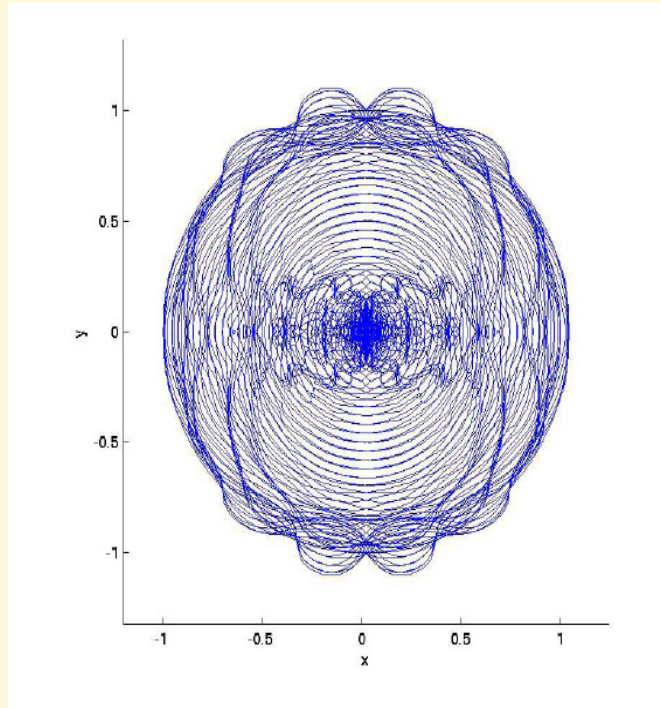


Figure 8: The projection of extremal surface in (x, y) -plane



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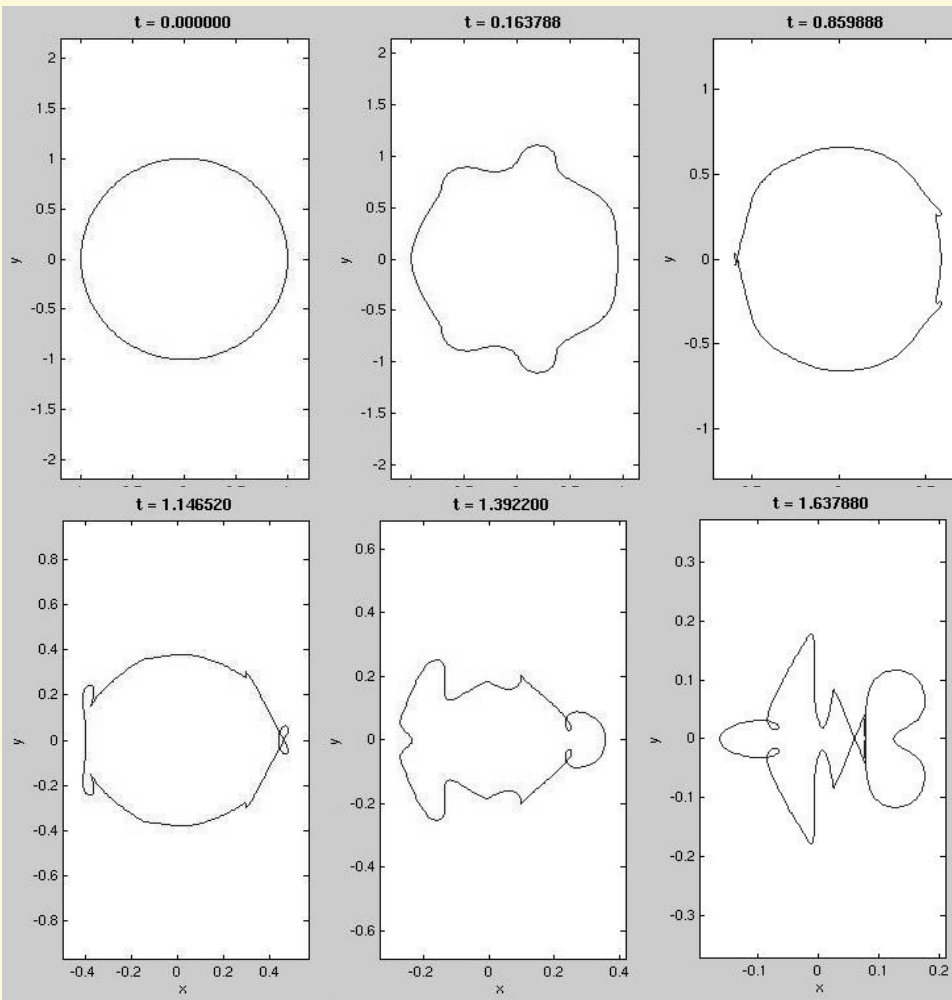
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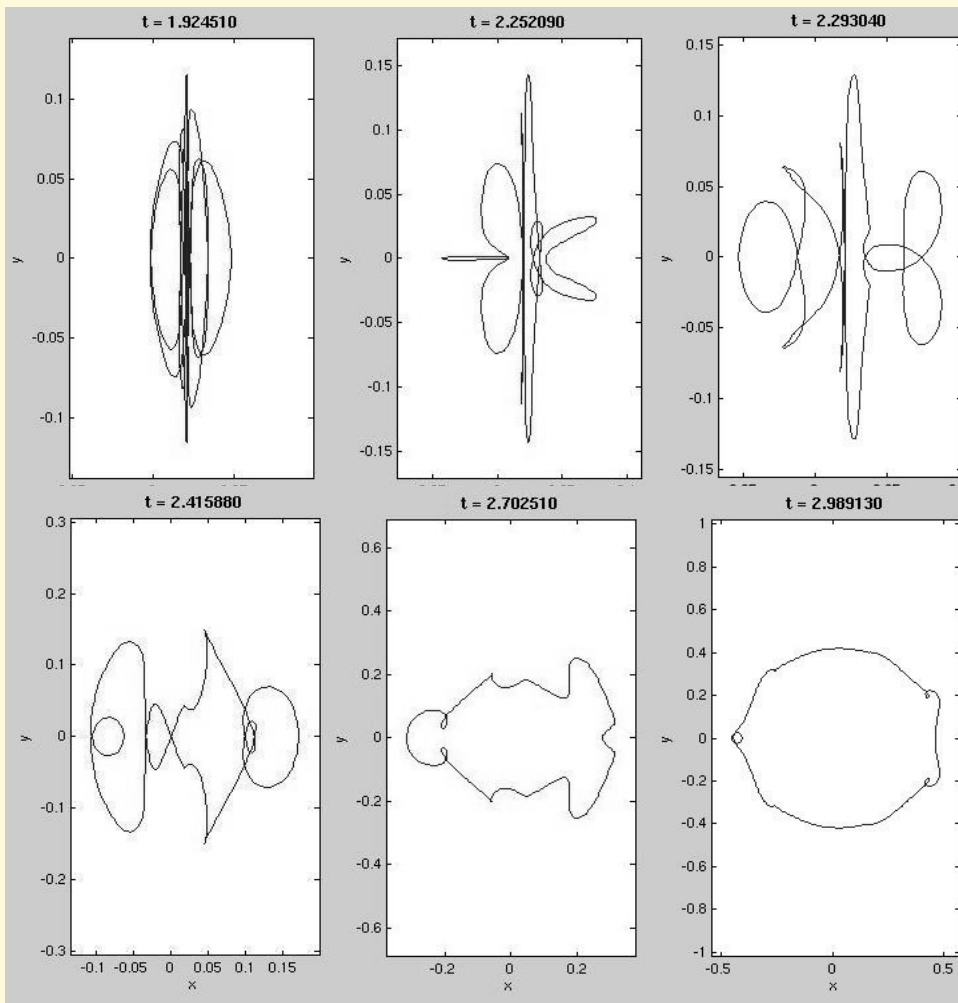
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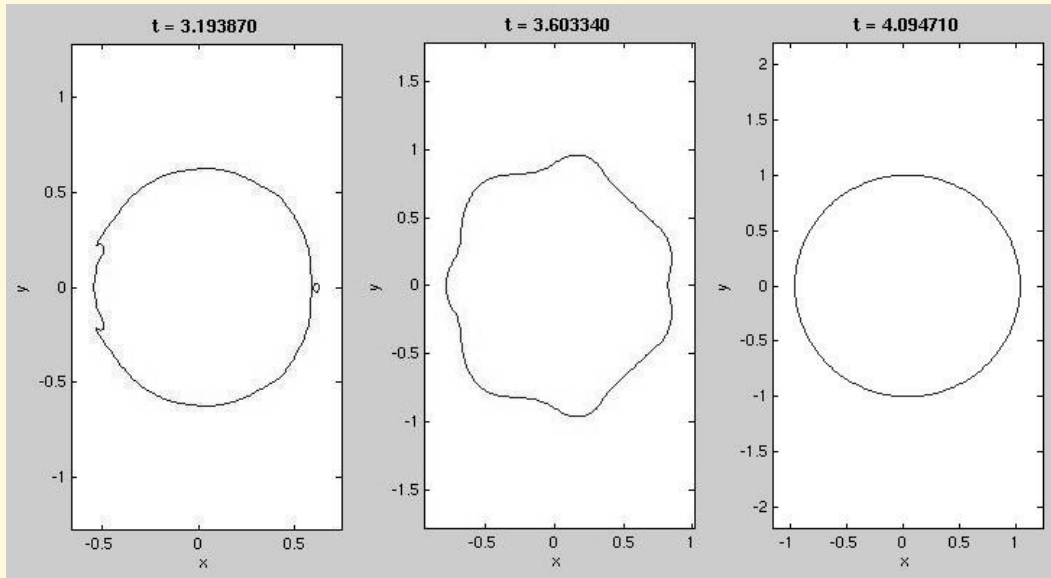


Figure 9: The dynamics of the closed string moving in \mathbb{R}^{1+2}

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See movies

From Figures and movies:

♣ In phase space, there are some topological singularities, even if the solution is still smooth. For example, there are some cross points in the string. In fact, for some cases, we can compute the number of these cross points, and their appearing time.

♣ Time periodicity!



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7. Time periodicity of the motion of closed strings

Consider a vector-valued function $f(t, \theta)$ defined on the domain $\mathbb{R}^+ \times \mathbb{R}$.

Definition 1 The function $f(t, \theta)$ is called to be **generalized time-periodic**, if there exists a positive constant T , a constant $\varphi(T)$ and a constant vector $\mathcal{D}(T)$ such that

$$f(t + T, \theta) = \mathcal{D}(T) + f(t, \theta + \varphi(T)), \quad \forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R},$$

where $\varphi(T)$ and $\mathcal{D}(T)$ depend only on T .

T : generalized period

$\varphi(T)$: generalized phase angle

$\mathcal{D}(T)$: translation displacement



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Let \mathcal{C}_t be the plane curve induced by the function $f(t, \theta)$.

Definition 2 The motion of the plane curve \mathcal{C}_0 is said to possess the **generalized time-periodicity**, if there exist a positive constant T and a translation mapping \mathcal{T} , depending on T but independent of t , such that

$$\mathcal{C}_{t+T} = \mathcal{T}\mathcal{C}_t,$$

namely,

$$\mathcal{T} : f(t, \cdot) \longrightarrow f(t + T, \cdot)$$

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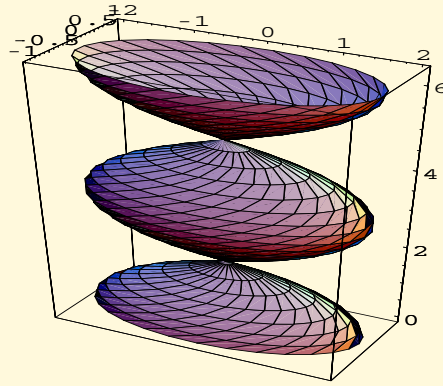


Figure 10: Generalized time-periodic function with $\mathcal{D}(T) = 0$

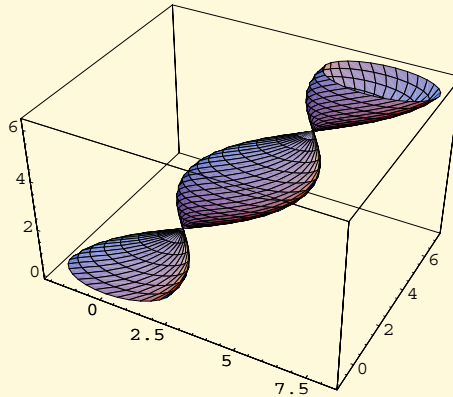


Figure 11: Generalized time-periodic function with $\mathcal{D}(T) \neq 0$



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Definition 3 The function $f(t, \theta)$ is called to be **essentially time-periodic**, if there exists a positive constant T , a constant vector $\mathcal{D}(T)$ and an unit orthogonal matrix $\mathcal{M}(T)$ such that, for every $t \in \mathbb{R}^+$

$$f(t + T, \cdot) = \mathcal{D}(T) + \mathcal{M}(T)f(t, \cdot), \quad (1)$$

where $\mathcal{D}(T)$ and $\mathcal{M}(T)$ depend only on T .

T : essential period

$\mathcal{D}(T)$: translation displacement

$\mathcal{M}(T)$: rotational factor

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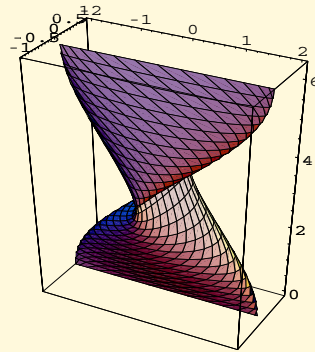


Figure 12: Essentially time-periodic function with rotation but without translation

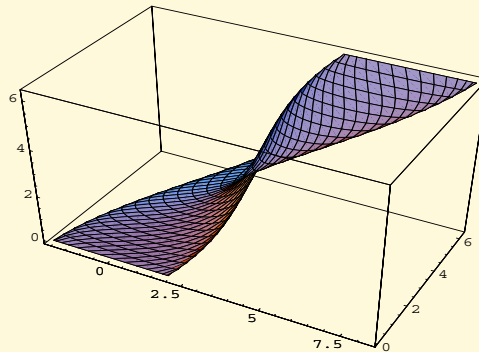


Figure 13: Essentially time-periodic function with rotation and translation



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Example Consider

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0, \infty) \times \mathbb{R}, \\ u = g, u_t = h & \text{on } \{t = 0\} \times \mathbb{R}, \end{cases}$$

where g, h are periodic functions with period, say, T .

$$u(t, x) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(\zeta) d\zeta.$$

It holds that

$$u(t+T, x) = u(t, x) + \int_0^T h(\zeta) d\zeta.$$

$u(t, x)$ is a generalized time-periodic function



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Assumption

$p(\theta), q(\theta)$: periodic functions with period \mathcal{P}

Theorem 1 (Global existence) Suppose that

$$\mathcal{L}(\theta) > 0, \quad \forall \theta \in [0, \mathcal{P}].$$

Then the Cauchy problem admits a unique global C^2 solution $x = x(t, \theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, if and only if,

$$\max_{\theta \in [0, \mathcal{P}]} \Lambda_-(\theta) < \min_{\theta \in [0, \mathcal{P}]} \Lambda_+(\theta).$$

Moreover, the solution $x = x(t, \theta)$ satisfies that, for arbitrary fixed $(t, \theta) \in \mathbb{R}^+ \times \mathbb{R}$, either $x_\theta(t, \theta) = 0$ or

$$\langle x_t(t, \theta), x_\theta(t, \theta) \rangle^2 - (|x_t(t, \theta)|^2 - 1)|x_\theta(t, \theta)|^2 > 0.$$

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Theorem 2 (Time-periodicity) Under the assumptions of Theorem 1, the solution $x = x(t, \theta)$ has a formula of d'Alembert's type, and possesses the generalized time-periodicity:

$$T = \int_0^{\mathcal{P}} \frac{1}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta,$$

$$\varphi_{\pm}(T) = - \int_0^{\mathcal{P}} \frac{\Lambda_{\pm}(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta,$$

$$\mathcal{D}_i(T) = \int_0^{\mathcal{P}} \mathcal{Q}_i(\zeta) d\zeta.$$

- ♣ The motion of closed strings possesses time periodicity
- ♣ The space periodicity implies the time periodicity


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The generalized phase angles are not unique:

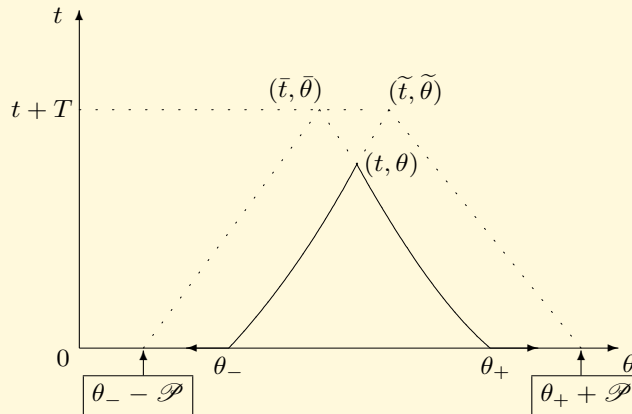


Figure 14: Translation of θ after θ_- or θ_+ moving one period



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Any fixed point θ^t in the string has two different moving ways. However, the resulting positions θ_{\pm}^{t+T} stand for the same point in the string at the time $t + T$:

$$\varphi_+(T) - \varphi_-(T) = -\mathcal{P},$$

then,

$$\theta_+^{t+T} - \theta_-^{t+T} = \mathcal{P}.$$

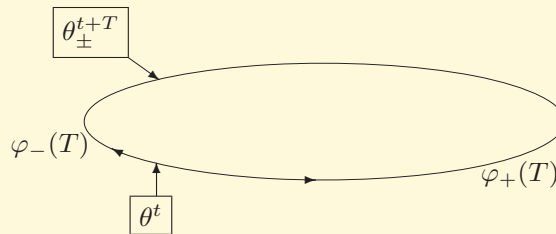


Figure 15: θ^t has two different moving ways, but the resulting positions stand for the same point



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8. Initial-boundary value problem for finite strings

- ♣ Some existence results
- ♣ Solution formula
- ♣ Several numerical examples

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Numerical examples

Consider the motion of an open string with the initial shape

$$x(0, \theta) = (\sin(2\pi\theta), \sin(4\pi\theta)), \quad \theta \in [0, 1],$$

the initial velocity

$$x_t(0, \theta) = (0, 0.5 \sin(2\pi\theta)), \quad \theta \in [0, 1]$$

and homogenous Dirichlet boundary conditions at their end points

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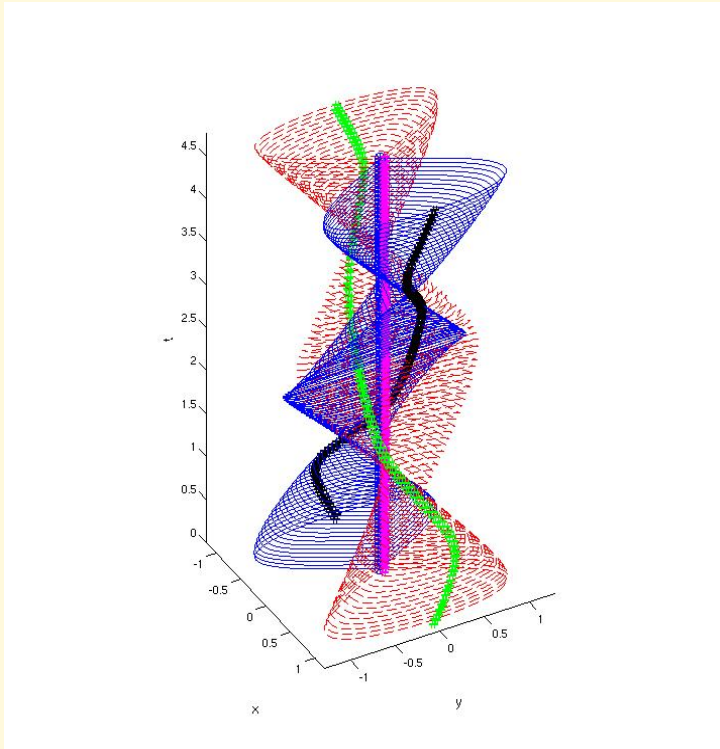


Figure 16: Extremal surface formed by the motion of open string



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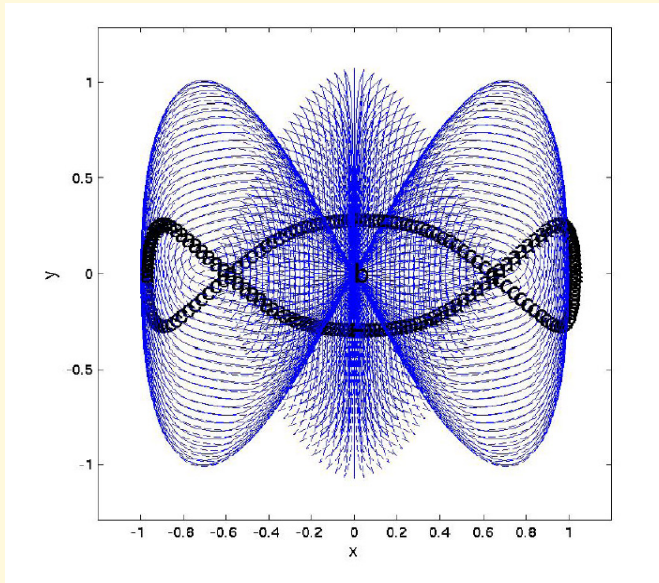


Figure 17: The projection of the above extremal surface

- See the movies for dynamics of motion of finite strings



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Open problems

- ◆ Extremal surfaces of mixed type?

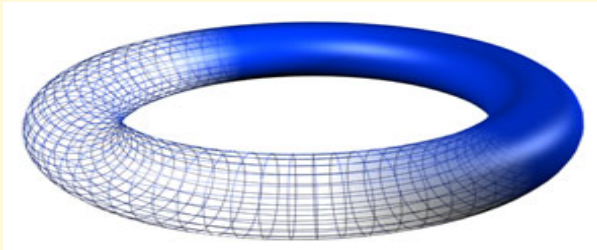
Singularity?

- ◆ The motion in a curved space-time?

Schwarzschild or Kerr Space-time?

- ◆ Multi-dimensional version?

The motion of a torus in \mathbb{R}^{1+3} ?



- Eigenvalues with constant multiplicity
- Linear degeneracy \oplus Symmetry



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Conjecture

The motion of any closed sub-manifold in the Minkowski space-time possesses the time periodicity

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Thank you



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